THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 6

Date: 25 October, 2024

1. (Exercise 3.5.2 of [BS11]) Show directly from the definition that the following are Cauchy sequences

   (a) $\left(\dfrac{n+1}{n}\right)$

   (b) $\left(\dfrac{1}{1!}+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right)$

2. Show directly that a bounded, monotone sequence is a Cauchy sequence without using the Monotone Convergence Theorem.
   Hint: by contradiction sps sequence is not Cauchy. Then show sequence cannot be bdd.

3. Show that the Archimedean Property and the Cauchy theorem together imply the axiom of completeness.

4. Use either the $\varepsilon-\delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\lim\limits_{x\to 0}\dfrac{x^2}{|x|}=0$

   (b) $\lim\limits_{x\to 1}\dfrac{x^2-x+1}{x+1}=\dfrac{1}{2}$

   (c) $\lim\limits_{x\to 0}\dfrac{1}{\sqrt{x}}$, $(x>0)$ does not exist.

5. (Exercise 4.1.14 of [BS11]) Suppose $c\in\mathbb{R}$ and f is a function on $\mathbb{R}$ so that $\lim\limits_{x\to c}(f(x))^2=L$.

   (a) If $L=0$, show that $\lim\limits_{x\to c}f(x)=0$ as well.

   (b) If $L\neq 0$, provide an example in which $\lim\limits_{x\to c}f(x)$ does not exist.

6. (Exercise 4.3.8 of [BS11]) Let f be defined on $(0,\infty)$ to $\mathbb{R}$. Prove that $\lim\limits_{x\to\infty}f(x)=L$ if and only if $\lim\limits_{x\to 0^+}f(1/x)=L$.

Announcement: Midterm next Wednesday 8:30am - 10:00am.

1. (Exercise 3.5.2 of [BS11]) Show directly from the definition that the following are Cauchy sequences

   (a) $\left(\dfrac{n+1}{n}\right)$

   (b) $\left(\dfrac{1}{1!}+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right)$

Pf: a): Let $\varepsilon>0$ be given. WLOG, can take $m>n$.

$\frac{n+1}{n} = 1+\frac{1}{n}$.

So $\left|1+\frac{1}{m}-1-\frac{1}{n}\right| = \left|\frac{1}{m}-\frac{1}{n}\right| \leq \frac{1}{m}+\frac{1}{n} \leq \frac{2}{n}$.

So taking $N(\varepsilon) > \frac{2}{\varepsilon}$ works.

b): Use $2^k < k!$ for $k \geq 4$.

2. Show directly that a bounded, monotone sequence is a Cauchy sequence without using the Monotone Convergence Theorem.

Pf: WLOG (x_n) is bounded above and monotone increasing.
Sps for contradiction (x_n) is not Cauchy.
So $\exists \varepsilon_0 > 0$ s.t. for all $N \in \mathbb{N}$, we can find $n > m \geq N$
s.t. $x_n - x_m \geq \varepsilon_0$. $\left(x_n - x_N \geq x_n - x_m \geq \varepsilon_0 . \right)$
By monotonicity, may as well take $m = N$.
Build a subsequence:
First, take $N = 1$. So there are $n_1 < n_2$ s.t.
$x_{n_2} - x_{n_1} \geq \varepsilon_0$.
For this n_2, take $n_3 > n_2$ s.t. $x_{n_3} - x_{n_2} \geq \varepsilon_0$.

$\vdots$

So we have x_{n_k} s.t. $x_{n_{k+1}} - x_{n_k} \geq \varepsilon_0$.

Take $M > x_1$. Then by AP., we can find $K \in \mathbb{N}$ s.t.
$K\varepsilon_0 > M - x_1$.

Then writing

$$x_{n_{K+1}} = x_1 + \sum_{k=1}^{K} (x_{n_{k+1}} - x_{n_k}) \geq x_1 + K\varepsilon_0 > M.$$

So (x_n) is unbounded, a contradiction. //

3. Show that the Archimedean Property and the Cauchy theorem together imply the axiom of completeness.

Rmk: Axiom of completeness $\Rightarrow$ MCT $\Rightarrow$ NIT $\Rightarrow$ BW

$\Rightarrow$ Cauchy Criterion $\underset{AP}{\Rightarrow}$ Completeness

Metric spaces / Banach spaces

"Completeness" iff all Cauchy sequences converge.

Pf: Let $A \subseteq \mathbb{R}$ that is non-empty and bdd. above.

Construct two sequences inductively.

Pick $a_1 \in A$, b_1 to be any u.b. of A.

consider, $\frac{1}{2}(a_1+b_1)$ - if this is an u.b. of A,

~~set~~ $a_2 = a_1$, $b_2 = \frac{1}{2}(a_1+b_1)$.

if not, set $a_2 = \frac{1}{2}(a_1+b_1)$, $b_2 = b_1$

$\vdots$

$a_k = \frac{1}{2}(a_{k-1}+b_{k-1})$ or a_{k-1}

$b_k = b_{k-1}$ or $\frac{1}{2}(a_{k-1}+b_{k-1})$

depending on whether $\frac{1}{2}(a_{k-1}+b_{k-1})$ is an u.b. of A.

By construction, (a_n), (b_n) are bdd, monotone,

hence by Q2, Cauchy $\Rightarrow$ converge.

Cauchy criterion

Moreover, $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \cdots = \frac{1}{2^{n-1}}(b_1 - a_1) \to 0$ as $n \to \infty$.

So set $x := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

WTS $x = \sup A$. Since $x = \lim_{n\to\infty} b_n$, and $b_n \geq a$ for all $n \in \mathbb{N}$, $a \in A$

so $x \geq a$ for all $a \in A$ (taking limits preserves order).

So x is an u.b. of A.

Let $\varepsilon > 0$. Then by $x = \lim_{n\to\infty} a_n$, $\exists N$ s.t. $\forall n \geq N$, $x - a_n < \varepsilon$.

$$\Updownarrow$$

$x - \varepsilon < a_n$.

So $x - \varepsilon$ is not an u.b. of A. So $x = \sup A$.

4. Use either the $\varepsilon - \delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\lim_{x \to 0} \frac{x^2}{|x|} = 0$

   (b) $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$

   (c) $\lim_{x \to 0} \frac{1}{\sqrt{x}}$, $(x > 0)$ does not exist.

Pf: b) Let $\varepsilon > 0$. Then

$$\left| \frac{x^2 - x + 1}{x+1} - \frac{1}{2} \right| = \left| \frac{(x-1)(2x-1)}{(2x+2)} \right| \leq |x-1| \left| \frac{2x-1}{2x+2} \right|$$

$$\leq |x-1| \left| \frac{\cancel{2x}}{\cancel{2x}} \right| = |x-1|.$$

So taking $\delta = \varepsilon$ works.

c) Use sequential criterion. Let $x_n = \frac{1}{n^2}$. Then $x_n \to 0$ as $n \to \infty$, and $x_n > 0$ for all $n \in \mathbb{N}$.

But we have

$$\frac{1}{\sqrt{x_n}} = \frac{1}{\sqrt{\frac{1}{n^2}}} = n$$ which diverges to infinity as $n \to \infty$.

5. (Exercise 4.1.14 of [BS11]) Suppose $c \in \mathbb{R}$ and f is a function on $\mathbb{R}$ so that $\lim\limits_{x \to c} (f(x))^2 = L$.

   (a) If $L = 0$, show that $\lim\limits_{x \to c} f(x) = 0$ as well.

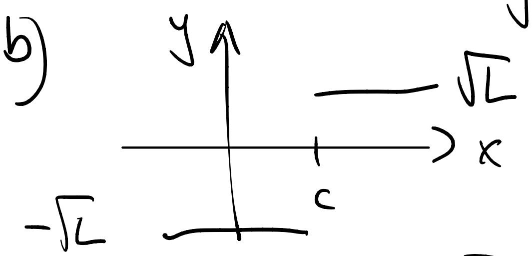
   (b) If $L \neq 0$, provide an example in which $\lim\limits_{x \to c} f(x)$ does not exist.

Pf: a) Sps $\lim\limits_{x \to c} (f(x))^2 = 0$. Then for all $\varepsilon > 0$, we can find $\delta > 0$ s.t. for $0 < |x - c| < \delta$, we have

$$|(f(x))^2| < \varepsilon^2$$

Taking square root yields the result.

b)



Take $f(x) = \begin{cases} \sqrt{L} & x \geq c \\ -\sqrt{L} & x < c. \end{cases}$

Then $(f(x))^2 \equiv L$ and $\lim\limits_{x \to c} (f(x))^2 = L$, but $\lim\limits_{x \to c} f(x)$ DNE.

6. (Exercise 4.3.8 of [BS11]) Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Prove that $\lim_{x \to \infty} f(x) = L$ if and only if $\lim_{x \to 0^+} f(1/x) = L$.

Pf: First sps $\lim_{x\to\infty} f(x) = L$. Then for all $\varepsilon > 0$, there is a $K(\varepsilon) > 0$ s.t. for any $x > K$

$$|f(x) - L| < \varepsilon.$$

Taking $\delta = \frac{1}{K}$, then we have for $0 < z < \frac{1}{K}$, we have $\frac{1}{z} > K$, and so $\left|f\left(\frac{1}{z}\right) - L\right| < \varepsilon$.

OTOH, sps, $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$. Then for all $\varepsilon > 0$, there is a $\delta(\varepsilon)$ s.t. for all $0 < x < \delta$,

$$\left|f\left(\tfrac{1}{x}\right) - L\right| < \varepsilon.$$

So by AP, taking $K > \frac{1}{\delta}$, we have that $0 < x < \delta \Rightarrow 0 < K < \frac{1}{x} =: z$ and

$$|f(z) - L| = \left|f\left(\tfrac{1}{x}\right) - L\right| < \varepsilon.$$

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